Diffusion in a one-dimensional random medium and hyperbolic Brownian motion

1996 J. Phys. A: Math. Gen. 29 1331

(http://iopscience.iop.org/0305-4470/29/7/006)

J. Phys. A: Math. Gen. **29** (1996) 1331–1345. Printed in the UK

Diffusion in a one-dimensional random medium and hyperbolic Brownian motion

Alain Comtet† and Cécile Monthus‡

Division de Physique Théorique§, IPN Bâtiment 100, 91406 Orsay, France
and
LPTPE, Université Paris 6, 4 Place Jussieu, 75252 Paris, France

Received 13 October 1995

Abstract. Classical diffusion in a random medium involves an exponential functional of Brownian motion. This functional also appears in the study of Brownian diffusion on a Riemann surface of constant negative curvature. We analyse in detail this relationship and study various distributions using stochastic calculus and functional integration.

1. Introduction

There is a close link between one-dimensional, or quasi-one-dimensional, disordered systems and Brownian diffusion on Riemann manifolds of constant negative curvature. Such a correspondence can be traced back to the pioneering work of Gertsenshtein and Vasil'ev [1] who have shown that the statistical properties of reflection and transmission coefficients for waveguides with random inhomogeneities are directly related to some random walk on the Lobachevsky plane. There has been renewed interest in this approach for the study of mesoscopic systems. The description of quasi-one-dimensional mesoscopic wires involves a Fokker–Planck equation for the probability distribution of the N eigenvalues of the transmission matrix [2]. Recently it has been shown that this equation can be interpreted as the diffusion equation on a Riemannian symmetric space [3]. An exact solution has been obtained in the unitary case by Beenaker and Rejaei [3]. Caselle then solved the general case [3] by relating it to a suitable Calogero–Sutherland model.

The purpose of this work is to show how the one-dimensional, classical diffusion of a particle in a quenched random potential $U(x)$, that is itself a Brownian motion, possibly with some constant drift, is directly related to Brownian motion on the hyperbolic plane. Since the latter is the archetype of chaotic systems [4], our work forms a bridge between disordered and chaotic systems.

† E-mail address: comtet@ipncls.in2p3.fr
‡ Present address: Service de Physique Théorique, C.E. Saclay, 91191 Gif-sur-Yvette, France. E-mail address: monthus@amoco.saclay.cea.fr
§ Unité de recherche des Universités Paris 11 et Paris 6 associée au CNRS.

0305-4470/96/071331+15$19.50

2. A fundamental random variable for one-dimensional, classical diffusion in a quenched random potential $U(x)$

Much work has been done on random walks, defined on a lattice by the master equation

$$P_n(t+1) = \alpha_{n-1} P_{n-1}(t) + \beta_{n+1} P_{n+1}(t) \tag{2.1}$$

where α_n is the random, quenched transition rate from site n to site $(n+1)$ and $\beta_n \equiv 1-\alpha_n$ is the random, quenched transition rate from site n to site $(n-1)$. From a physical point of view it is convenient to introduce a corresponding random potential $U(n)$ on each site n and to write the ratio of the two transition rates α_n and β_n as an Arrhenius factor

$$\sigma_n \equiv \frac{\beta_n}{\alpha_n} = \frac{\mathrm{e}^{-\beta[U(n-1)-U(n)]}}{\mathrm{e}^{-\beta[U(n+1)-U(n)]}} = \mathrm{e}^{\beta[U(n+1)-U(n-1)]}. \tag{2.2}$$

The study of different physical quantities related to this random walk [5] involves systematically random variables of the form

$$Z(a,b) = \sum_{n=a}^{b} \prod_{k=a}^{n} \sigma_k = \sigma_a + \sigma_a\sigma_{a+1} + \cdots + \sigma_a\sigma_{a+1}\ldots\sigma_b. \tag{2.3}$$

The fundamental property of these variables is to satisfy the linear, random coefficient, recurrence relation

$$Z(a,b) = \sigma_a[1 + Z(a+1,b)]. \tag{2.4}$$

The Ising chain in a random magnetic field [6] also involves such a discrete multiplicative stochastic process.

Let us now consider the one-dimensional continuous model of classical diffusion defined by the Fokker–Planck equation for the probability density $P(x,t|x_0,0)$

$$\frac{\partial P}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} - \beta F(x) P\right) \tag{2.5}$$

where $\{F(x)\}$ is a quenched random force. In this continuous limit, the discrete random variable $Z(a,b)$ defined in (2.3) becomes an exponential functional of the random potential $U(x) = -\int^x F(y)\,\mathrm{d}y$

$$\tau(a,b) = \int_a^b \mathrm{d}x\ \mathrm{e}^{[\beta[U(x)-U(a)]} = \int_a^b \mathrm{d}x\ \exp\left[-\beta\int_a^x F(y)\,\mathrm{d}y\right]. \tag{2.6}$$

The evolution of this functional is governed by the stochastic differential equation

$$\frac{\partial\tau}{\partial a}(a,b) = \beta F(a)\tau(a,b) - 1 \tag{2.7}$$

which replaces the random-coefficient recurrence relation (2.4) satisfied by $Z(a,b)$. Note that the stochastic term $F(a)$ appears multiplicatively, so that the fluctuations of the random force are coupled to the values taken by the random process $\tau(a,b)$.

We shall now explain how the functional $\tau(a,b)$ arises in some physical quantities associated with the classical diffusion of a particle in a quenched, random environment.

- If the random force has a positive mean $\langle F(x)\rangle \equiv F_0 > 0$, then the probability distribution of the random functional

$$\tau_\infty \equiv \lim_{(b-a)\to\infty} \tau(a,b) \tag{2.8}$$

determines the large-time, anomalous behaviour of the position of the Brownian particle [7]. In particular the velocity defined for each sample as

$$V = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{+\infty} \mathrm{d}x\, x\, P(x, t|x_0, 0) \tag{2.9}$$

is a self-averaging quantity inversely proportional to the first moment of τ_∞ [7]

$$V = \frac{1}{2\langle\tau_\infty\rangle}. \tag{2.10}$$

When the quenched random force is distributed with the Gaussian measure

$$\mathcal{D}F(x) \exp\left[-\frac{1}{2\sigma}\int \mathrm{d}x\,(F(x) - F_0)^2\right] \tag{2.11}$$

then the probability distribution $\mathcal{P}_\infty(\tau)$ of the functional τ_∞ is [7]

$$\mathcal{P}_\infty(\tau) = \frac{\alpha}{\Gamma(\mu)}\left(\frac{1}{\alpha\tau}\right)^{1+\mu} \mathrm{e}^{-1/\alpha\tau} \underset{\tau\to\infty}{\propto} \frac{1}{\tau^{1+\mu}} \tag{2.12}$$

where $\mu = 2F_0/\beta\sigma > 0$ is a dimensionless parameter and $\alpha = \sigma\beta^2/2$. This algebraic decay for large τ explains the dynamical phase transitions that occur in this model [7]. In particular equation (2.10) implies that the value $\mu = 1$ separates a phase of vanishing velocity $V = 0$ for $0 < \mu < 1$ and a phase of finite velocity $V > 0$ for $\mu > 1$.

• The functional $\tau(a, b)$ also arises in the study of the transport properties of finite-size disordered samples. The stationary current J_N which goes through a disordered sample of length N with fixed concentrations P_0 and P_N at the boundary can be written in terms of the exponential functional $\tau_N \equiv \tau(0, N)$ as [8, 9]

$$J_N = \frac{1}{2}\left[\frac{P_0}{\tau_N} - P_N \frac{\partial \ln \tau_N}{\partial N}\right]. \tag{2.13}$$

When the end $x = N$ is a trap described by the boundary condition $P_N = 0$, the flux J_N is simply a random variable inversely proportional to τ_N. The probability distribution of τ_N has been studied for the case of zero mean force $F_0 = 0$ [8] and for the general case with arbitrary mean force [9] by different methods.

The functional τ_N has also been applied to problems of finance [10–12]. Yor has pointed out [11] the relation between the functional τ_N for the particular case of $\mu = \frac{1}{2}$, and free Brownian motion on the hyperbolic plane. In the following, we first rederive this correspondence and then generalize it to arbitrary μ, using an external drift on the hyperbolic plane.

3. Relation to hyperbolic Brownian motion

The upper half-plane $\{(x, y), y > 0\}$ endowed with the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \tag{3.1}$$

defines a two-dimensional Riemann manifold of constant negative Gaussian curvature $R = -1$. The surface element $\mathrm{d}S$ and the Laplace operator Δ are covariantly defined as

$$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \qquad \text{and} \qquad \Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{3.2}$$

Free Brownian motion on this manifold is defined by the diffusion equation for the Green function $G_t(x, y)$

$$\frac{\partial G}{\partial t} = D\Delta G = Dy^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)G. \tag{3.3}$$

It is convenient to choose the initial condition at the point $(x = 0, y = 1)$

$$G_t(x, y) \underset{t\to 0^+}{\longrightarrow} \delta(x)\delta(y-1). \tag{3.4}$$

The normalization of the Green function $G_t(x, y)$ then reads for any time t

$$1 = \int \mathrm{d}S\, G_t(x, y) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_0^{+\infty} \mathrm{d}y\, \frac{1}{y^2}\, G_t(x, y). \tag{3.5}$$

Consider the probability density $P_t(x, y)$

$$P_t(x, y) = \frac{1}{y^2} G_t(x, y) \tag{3.6}$$

normalized with respect to the flat measure $\mathrm{d}x\,\mathrm{d}y$

$$1 = \int_{-\infty}^{+\infty} \mathrm{d}x \int_0^{+\infty} \mathrm{d}y\, P_t(x, y). \tag{3.7}$$

This probability density satisfies the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) y^2 P \tag{3.8}$$

and the initial condition

$$P_t(x, y) \underset{t\to 0^+}{\longrightarrow} \delta(x)\delta(y-1). \tag{3.9}$$

We now introduce two independant Gaussian white noises $\eta_1(t)$ and $\eta_2(t)$ and write the stochastic differential equations for the process $\{x(t), y(t)\}$ corresponding to the Fokker–Planck equation (3.8) following respectively the Itô or Stratonovich convention [13]

$$\text{Itô} \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2D}y\eta_1(t) \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{2D}y\eta_2(t) \end{cases} \tag{3.10}$$

$$\text{Stratonovich} \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2D}y\eta_1(t) \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} = -Dy + \sqrt{2D}y\eta_2(t). \end{cases} \tag{3.11}$$

Direct integration of the stochastic differential equation for the process $y(t)$ gives immediately

$$y(t) = \exp\left[-Dt + \sqrt{2D}\int_0^t \eta_2(s)\,\mathrm{d}s\right]. \tag{3.12}$$

The process $y(t)$ is therefore simply the exponential of a linear Brownian motion with negative drift. Let us now study more precisely the process $x(t)$ which evolves according to the Langevin equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2D}y(t)\eta_1(t) \tag{3.13}$$

where $y(t)$ is a process statistically independent of the white noise $\eta_1(t)$. It is convenient to look for a change $\tau(t)$ of the time t which transforms this equation into the Langevin equation for linear Brownian motion in this new time $\tau(t)$:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \sqrt{2D}\eta(\tau). \tag{3.14}$$

By using the homogeneity properties of the white noise

$$\eta(\tau(t)) = \frac{\eta_1(t)}{\sqrt{\mathrm{d}\tau/\mathrm{d}t}} \tag{3.15}$$

one may determine the time transformation necessary to pass from (3.13) to (3.14):

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = y^2(t). \tag{3.16}$$

We thus obtain the representation given by Yor [11]

$$x(t) = \sqrt{2D}\int_0^{\tau_t} \mathrm{d}u\, \eta(u) \tag{3.17}$$

with

$$\tau_t = \int_0^t \mathrm{d}v\, y^2(v). \tag{3.18}$$

More explicitly by using (3.12) one has

$$\tau_t = \int_0^t \mathrm{d}v \exp\left[-2Dv + 2\sqrt{2D}\int_0^v \eta_2(s)\,\mathrm{d}s\right]. \tag{3.19}$$

If we set

$$\beta\int_0^v F(u)\,\mathrm{d}u = 2Dv - 2\sqrt{2D}\int_0^v \eta_2(s)\,\mathrm{d}s \tag{3.20}$$

then the new time τ_t is nothing but the functional (2.6)

$$\tau_t = \int_0^t \mathrm{d}v\, \exp\left[-\beta\int_0^v F(u)\,\mathrm{d}u\right] \tag{3.21}$$

which is encountered in the study of classical diffusion in a quenched force $\{F\}$ distributed as the Gaussian white noise

$$\mathcal{D}F(x)\exp\left[-\frac{1}{2\sigma}\int_{-\infty}^{+\infty}(F(x)-F_0)^2\,\mathrm{d}x\right] \tag{3.22}$$

with parameters $F_0 = 2D/\beta$ and $\sigma = 8(D/\beta^2)$.

In other words, the free Brownian motion $\{x(t), y(t)\}$ on the hyperbolic plane can be rewritten in terms of two independent white noises of measure

$$\mathcal{D}F(x)\exp\left[-\frac{1}{2\sigma}\int_{-\infty}^{+\infty}(F(x)-F_0)^2\,\mathrm{d}x\right] \qquad \text{and} \qquad \mathcal{D}\eta(t)\exp\left[-\frac{1}{2}\int \mathrm{d}t\, \eta^2(t)\right]. \tag{3.23}$$

The process $y(t)$ is simply the exponential of the Brownian motion with drift $U(t) = -\int_0^t F(u)\,\mathrm{d}u$:

$$y(t) = \exp\left[-\frac{\beta}{2}\int_0^t F(u)\,\mathrm{d}u\right] = \mathrm{e}^{\beta U(t)/2}. \tag{3.24}$$

The process $x(t)$ can be viewed as a linear Brownian motion

$$x(t) = \sqrt{2D} \int_0^{\tau_t} \mathrm{d}u\, \eta(u) \tag{3.25}$$

with an effective time τ_t that is itself a random process depending on $y(t)$

$$\tau_t = \int_0^t \mathrm{d}v\, \mathrm{e}^{\beta U(v)} = \int_0^t \mathrm{d}v\, y^2(v). \tag{3.26}$$

Note that the dimensionless parameter $\mu \equiv 2F_0/\beta\sigma$ which characterizes the different phases of anomalous diffusion is $\frac{1}{2}$ for free hyperbolic Brownian motion. This value represents the natural drift induced by the curvature of the Poincaré half-plane. In fact one may generalize this analysis to arbitrary μ by introducting an external constant drift m in the direction y:

$$\mu = \tfrac{1}{2} + m. \tag{3.27}$$

The corresponding stochastic differential equations then read

$$\text{Itô} \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2D}\, y\eta_1(t) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -2Dmy + \sqrt{2D}\, y\eta_2(t) \end{cases} \tag{3.28}$$

$$\text{Stratonovich} \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2D}\, y\eta_1(t) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -2D\mu y + \sqrt{2D}\, y\eta_2(t). \end{cases} \tag{3.29}$$

For any μ there is therefore a direct correspondence through equations (3.24)–(3.26) between the joint stochastic process characterizing the one-dimensional diffusion {random potential $U(t)$, exponential functional τ_t} and the Brownian motion $\{x(t), y(t)\}$ on the hyperbolic plane with possibly some external constant drift m along direction y. We now consider some consequences of this correspondence.

4. Marginal laws of the processes τ_t, $x(t)$ and $y(t)$

The marginal law $Y_t(y)$ of the process $y(t)$ reads according to equation (3.24)

$$Y_t(y) = \int_{U(0)=0} \mathcal{D}U(s) \exp\left[-\frac{1}{2\sigma}\int_0^t \left(\frac{\mathrm{d}U}{\mathrm{d}s} + F_0\right)^2 \mathrm{d}s\right] \delta(y - \mathrm{e}^{\beta U(t)/2}). \tag{4.1}$$

We thus get after some algebra the following log-normal distribution

$$Y_t(y) = \frac{1}{y\sqrt{4\pi Dt}} \exp\left[-\frac{1}{4Dt}(\ln(y) + 2\mu Dt)^2\right] \tag{4.2}$$

where $\mu = 2F_0/\beta\sigma$ and $D = \beta^2\sigma/8$. In the case of free Brownian motion ($\mu = \frac{1}{2}$), this marginal law tends to a δ distribution in the limit $t \to \infty$

$$Y_\infty(y) = \delta(y). \tag{4.3}$$

The Brownian particle is therefore attracted to the $y = 0$ axis as a result of the curvature of the hyperbolic plane. Note that this axis represents infinity on this plane. This limit law remains unchanged as long as $\mu \equiv (\frac{1}{2} + m) > 0$. However, when the constant external drift m in the y direction is negative enough to overcome the natural drift of the hyperbolic plane ($m < -\frac{1}{2}$), there is no equilibrium distribution for the process $y(t)$. We shall now show how in the case $\mu > 0$ the existence of the stationary distribution (4.3) for the process $y(t)$ may govern the existence of a stationary distribution for the process $x(t)$.

The process $x(t)$ is a Brownian motion of effective time τ_t which is a functional of the process $y(t)$:

$$x(t) = \sqrt{2D}\int_0^{\tau_t} \mathrm{d}u\,\eta(u) \qquad \text{with } \tau_t = \int_0^t \mathrm{d}v\,\mathrm{e}^{\beta U(v)} = \int_0^t \mathrm{d}v\,y^2(v). \quad (4.4)$$

The statistical independence of $\eta(t)$ and τ_t allows us to write the marginal law $X_t(x)$ of the process $x(t)$ in terms of the probability distribution $\psi_t(\tau)$ of the functional τ_t as

$$X_t(x) = \int_0^\infty \mathrm{d}\tau\;\psi_t(\tau)\frac{1}{\sqrt{4\pi D\tau}}\mathrm{e}^{-x^2/4D\tau}. \quad (4.5)$$

In a previous work [9], we have shown that $\psi_t(\tau)$ obeys a Fokker–Planck equation whose solution can be expressed in terms of an expansion on a suitable eigenvector basis. Let us rederive briefly this expansion for completeness. The starting point is the Langevin equation (2.7) satisfied by $\tau(a,b)$ defined in (2.6):

$$\frac{\partial\tau}{\partial a}(a,b) = \beta F(a)\tau(a,b) - 1 \quad (4.6)$$

where the random force F is Gaussian white noise of measure

$$\mathcal{D}F(x)\exp\left[-\frac{1}{2\sigma}\int_{-\infty}^{+\infty}(F(x)-F_0)^2\,\mathrm{d}x\right]. \quad (4.7)$$

The process $\tau(a,b)$ is therefore a multiplicative Markov process. By using the Stratonovich prescription for (4.6), one may write the corresponding Fokker–Planck equation for the probability distribution $P_{a,b}(\tau)$ as

$$-\frac{\partial P_{a,b}(\tau)}{\partial a} = \frac{\partial}{\partial\tau}\left[\alpha\tau^2\frac{\partial P_{a,b}(\tau)}{\partial\tau} + ((\mu+1)\alpha\tau - 1)P_{a,b}(\tau)\right] \quad (4.8)$$

which must be supplemented by the initial condition $P_{b,b}(\tau) = \delta(\tau)$.

Since $P_{a,b}(\tau)$ can only depend on the length $(b-a)$ of the integration domain defining $\tau(a,b)$ in equation (2.6), we have

$$P_{a,b}(\tau) = \psi_t(\tau) \qquad \text{where } t = b-a. \quad (4.9)$$

The probability distribution $\psi_t(\tau)$ of the process τ_t therefore satisfies the Fokker–Planck equation

$$\frac{\partial\psi_t(\tau)}{\partial t} = \frac{\partial}{\partial\tau}\left[\alpha\tau^2\frac{\partial\psi_t(\tau)}{\partial\tau} + ((\mu+1)\alpha\tau - 1)\psi_t(\tau)\right] \quad (4.10)$$

with the initial condition $\psi_{t=0}(\tau) = \delta(\tau)$.

We now recall the solution [9] expanded in the Fokker–Planck eigenvector basis

$$\begin{aligned}\psi_t(\tau) = \alpha &\sum_{0\leqslant n<\mu/2}\mathrm{e}^{-\alpha t n(\mu-n)}\frac{(-1)^n(\mu-2n)}{\Gamma(1+\mu-n)}\left(\frac{1}{\alpha\tau}\right)^{1+\mu-n} L_n^{\mu-2n}\left(\frac{1}{\alpha\tau}\right)\mathrm{e}^{-1/\alpha\tau}\\ &+\frac{\alpha}{4\pi^2}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\alpha t(\mu^2+s^2)/4}s\sinh\pi s\left|\Gamma\left(-\frac{\mu}{2}+\mathrm{i}\frac{s}{2}\right)\right|^2\left(\frac{1}{\alpha\tau}\right)^{(1+\mu)/2}\\ &\times W_{(1+\mu)/2,\mathrm{i}s/2}\left(\frac{1}{\alpha\tau}\right)\mathrm{e}^{-1/2\alpha\tau}\end{aligned} \quad (4.11)$$

where the L_n^μ are Laguerre polynomials and the $W_{\mu,\nu}$ Whittaker functions. By integrating with the Gaussian kernel (4.5) and by using the transposition $\alpha = 4D$, one may obtain an expansion presenting the same time-relaxation spectrum

$$X_t(x) = \sum_{0\leqslant n<\mu/2} \mathrm{e}^{-4Dn(\mu-n)t} \frac{(-1)^n(\mu-2n)}{n!\Gamma(1+\mu-n)} \frac{\Gamma(\mu+\frac{1}{2}-n)}{\Gamma(\frac{1}{2}-n)} \left(\frac{1}{1+x^2}\right)^{\mu+\frac{1}{2}} \times F(-n, n-\mu, \tfrac{1}{2}; -x^2) + \frac{1}{4\pi^2\sqrt{\pi}} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-D(\mu^2+s^2)t} s \sinh \pi s \left|\Gamma\left(-\frac{\mu}{2}+\mathrm{i}\frac{s}{2}\right)\right|^2 \left|\Gamma\left(\frac{\mu+1}{2}+\mathrm{i}\frac{s}{2}\right)\right|^2 \times F\left(\frac{\mu+1}{2}+\mathrm{i}\frac{s}{2}, \frac{\mu+1}{2}-\mathrm{i}\frac{s}{2}, \frac{1}{2}; -x^2\right) \tag{4.12}$$

where $F(a, b, c; z)$ denotes the hypergeometric function of parameters (a, b, c).

We note parenthetically that the marginal law $X_t(x)$ itself satisfies a Fokker–Planck equation. Indeed equations (4.5) and (4.10) lead directly to

$$\frac{\partial X_t(x)}{\partial t} = D\frac{\partial}{\partial x}\left[(1+x^2)\frac{\partial X_t(x)}{\partial x} + (2\mu+1)xX_t(x)\right]. \tag{4.13}$$

To our knowledge, this equation which has been obtained here through the link (3.17) with the functional τ_t, has never been obtained directly.

Let us stress the utility of the developments (4.11) and (4.12) in the study of long-time behaviour. For $\mu < 0$ the relaxation spectrum is purely continuous, and there is no limit distribution. However, for $\mu > 0$ there is at least one discrete term ($n = 0$) that corresponds to an equilibrium distribution $X_\infty(x)$ in the limit $t \to \infty$:

$$X_\infty(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu)} \left(\frac{1}{1+x^2}\right)^{\mu+\frac{1}{2}}. \tag{4.14}$$

For $0 < \mu < 2$ this is the only discrete term, and the relaxation towards equilibrium is thus entirely governed by the continuum. For $2 < \mu$ there are other discrete terms, for example $n = 1$, that govern the exponential relaxation towards equilibrium.

The existence of the limit law (4.14) for hyperbolic Brownian motion reveals a 'localization' phenomenon in direction x. This effect comes from the attraction towards the axis $y = 0$. Note that for the free case ($\mu = \frac{1}{2}$), the asymptotic marginal law $X_\infty(x)$ is simply a Lorentzian:

$$X_\infty(x) = \frac{1}{\pi}\frac{1}{1+x^2}. \tag{4.15}$$

In sum there exist equilibrium distributions for Y_∞ and X_∞ as long as $\mu = (\frac{1}{2}+m) > 0$:

$$\begin{cases} Y_\infty(y) = \delta(y) \\ X_\infty(x) = \dfrac{1}{\sqrt{\pi}} \dfrac{\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu)} \left(\dfrac{1}{1+x^2}\right)^{\mu+\frac{1}{2}}. \end{cases} \tag{4.16}$$

The joint law of the processes $\{x(t), y(t)\}$ thus exhibits the factorization form in the limit $t \to \infty$:

$$P_\infty(x, y) = X_\infty(x)\delta(y). \tag{4.17}$$

However, as long as time t is finite, the two processes $x(t)$ and $y(t)$ remain coupled. The study of their joint law is then needed to get a complete description of hyperbolic Brownian motion.

5. Joint laws of the processes $\{\tau_t, U(t)\}$ and $\{x(t), y(t)\}$

By using path integrals, we may obtain very simply the joint law $\psi_t(\tau \parallel u)$ of the random variables

$$\tau_t = \int_0^t \mathrm{d}x\, \mathrm{e}^{\beta U(x)} \qquad \text{and} \qquad U(t).$$

Let us generalize the approach we used in [9]. The τ-Laplace transform of the joint law

$$E(\mathrm{e}^{-p\tau_t} \parallel u) \equiv \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-p\tau} \psi_t(\tau \parallel u) \tag{5.1}$$

can be written as a path integral over the random potential

$$\begin{aligned} E(\mathrm{e}^{-p\tau_t} \parallel u) &= \int_{U(0)=0}^{U(t)=u} \mathcal{D}U(x) \exp\left[-\frac{1}{2\sigma} \int_0^t \left(\frac{\mathrm{d}U}{\mathrm{d}x} + F_0 \right)^2 \mathrm{d}x - p \int_0^t \mathrm{d}x\, \mathrm{e}^{\beta U(x)} \right] \\ &= \exp(-F_0^2 t/2\sigma) \exp(-F_0 u/\sigma) \int_{U(0)=0}^{U(t)=u} \mathcal{D}U(x) \\ &\quad \times \exp\left[-\frac{1}{2\sigma} \int_0^t \left(\frac{\mathrm{d}U}{\mathrm{d}x} \right)^2 \mathrm{d}x - p \int_0^t \mathrm{d}x\, \mathrm{e}^{\beta U(x)} \right]. \end{aligned} \tag{5.2}$$

The remaining path integral is simply the Euclidean quantum-mechanical Green function $\langle u|\mathrm{e}^{-tH}|0\rangle$ associated with the Liouville Hamiltonian

$$H = -\frac{\sigma}{2} \frac{\mathrm{d}^2}{\mathrm{d}u^2} + p\mathrm{e}^{\beta u}. \tag{5.3}$$

We therefore get

$$E(\mathrm{e}^{-p\tau_t} \parallel u) = \mathrm{e}^{-F_0^2 t/2\sigma} \mathrm{e}^{-F_0 u/\sigma} \langle u|\mathrm{e}^{-tH}|0\rangle. \tag{5.4}$$

The expansion of the Green function $\langle u|\mathrm{e}^{-tH}|0\rangle$ in the basis of eigenfunctions $\psi_k(u)$

$$\psi_k(u) = 2\sqrt{\frac{\beta k}{\alpha\pi} \sinh \frac{2k\pi}{\sqrt{\alpha}}}\, K_{2\mathrm{i}k/\sqrt{\alpha}} \left(2\sqrt{\frac{p}{\alpha}} \mathrm{e}^{\beta u/2} \right) \tag{5.5}$$

gives

$$\langle u|\mathrm{e}^{-tH}|0\rangle = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \psi_k(u) \psi_k^*(0) \mathrm{e}^{-k^2 t}. \tag{5.6}$$

We finally obtain as the τ-Laplace transform of the joint law $\phi_t(\tau \parallel y)$ of the random variables

$$\tau_t = \int_0^t \mathrm{d}x\, \mathrm{e}^{\beta U(x)} \qquad \text{and} \qquad y(t) = \mathrm{e}^{\beta U(t)/2}$$

after some changes of variables

$$\begin{aligned} E(\mathrm{e}^{-p\tau_t} \parallel y) &\equiv \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-p\tau} \phi_t(\tau \parallel y) \\ &= \frac{\mathrm{e}^{-\alpha t \mu^2/4}}{\pi^2} \frac{1}{y^{1+\mu}} \int_{-\infty}^{+\infty} \mathrm{d}q\, \mathrm{e}^{-\alpha t q^2/4} q \sinh \pi q\, K_{\mathrm{i}q} \left(2y\sqrt{\frac{p}{\alpha}} \right) K_{\mathrm{i}q} \left(2\sqrt{\frac{p}{\alpha}} \right). \end{aligned} \tag{5.7}$$

We now compute the x^2-Laplace transform of the joint law $Q_t(x, y)$ of the hyperbolic Brownian motion starting from the series of moments of $x^2(t)$ with $y(t)$ being fixed as the constant y

$$
\begin{aligned}
E(\mathrm{e}^{-qx^2(t)} \parallel y) &\equiv \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-qx^2} Q_t(x, y) \\
&= \sum_{n=0}^{\infty} \frac{(-q)^n}{n!} E(x^{2n}(t) \parallel y(t) = y).
\end{aligned}
\tag{5.8}
$$

By using equation (3.17), we may obtain a relation between the moments of $x^2(t)$ and the moments of τ_t, when $y(t)$ is fixed as y:

$$
E(x^{2n}(t) \parallel y) = (2D)^n \mathcal{N}(n) E(\tau_t^n(t) \parallel y)
\tag{5.9}
$$

where $\mathcal{N}(n)$ denotes the Wick combinatorial factor that counts the number of ways to pair the $2n$ functions η. In particular $\mathcal{N}(n)$ is equal to the moment of order $2n$ of a suitable Gaussian random variable ξ with variance unity:

$$
\mathcal{N}(n) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{\sqrt{2\pi}} \xi^{2n} \mathrm{e}^{-\xi^2/2} = 2^n \frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})} = \prod_{k=1}^{n} (2k - 1).
\tag{5.10}
$$

By using this integral representation, we can resum the series of moments of τ_t under the integral

$$
E(\mathrm{e}^{-qx^2(t)} \parallel y) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{\sqrt{2\pi}} \mathrm{e}^{-\xi^2/2} E(\mathrm{e}^{-q2D\xi^2\tau_t} \parallel y).
\tag{5.11}
$$

From equation (5.7) and the correspondence $\alpha = 4D$, we get

$$
\begin{aligned}
E(\mathrm{e}^{-qx^2(t)} \parallel y) = {}& \frac{1}{\pi^2 \sqrt{\pi q}} \left(\frac{1}{y}\right)^{\mu+1} \int_0^{\infty} \mathrm{d}k\, \mathrm{e}^{-k^2/4q} \\
& \times \int_{-\infty}^{+\infty} \mathrm{d}\nu\, \mathrm{e}^{-Dt(\mu^2+\nu^2)} \nu \sinh \pi\nu K_{\mathrm{i}\nu}(ky) K_{\mathrm{i}\nu}(k).
\end{aligned}
\tag{5.12}
$$

For the free case ($\mu = \frac{1}{2}$), one may also obtain this expression from the Green function $G_t(x, y)$ on the hyperbolic plane [14]:

$$
E(\mathrm{e}^{-qx^2(t)} \parallel y) = \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{-qx^2} \frac{1}{y^2} G_t(x, y).
\tag{5.13}
$$

We finally mention that an alternative form of the joint law $\phi_t(\tau \parallel y)$ given in equation (5.7) has been obtained by Yor [12] through the time Laplace transform

$$
\int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-st} \phi_t(\tau \parallel y) = \frac{1}{\tau y^{1+\mu}} \mathrm{e}^{-((1+z^2)/\alpha\tau)} I_\nu \left(\frac{z}{\alpha\tau}\right) \qquad \text{where } \nu = \sqrt{\mu^2 + 4\frac{s}{\alpha}}.
\tag{5.14}
$$

One may also find in the mathematical literature [10–12] different expressions related to the probability distributions of the functional $\tau\{T_s\}$, where T_s is an independent time, exponentially distributed with parameter s.

6. Conformal mapping from Poincaré half-plane to unit disk

Let us consider the conformal mapping w from the Poincaré upper half-plane $\{z = x + \mathrm{i}y,\ y > 0\}$ to the unit disk $\{w = r\mathrm{e}^{\mathrm{i}\theta},\ |r| \leqslant 1\}$:

$$w = \frac{\mathrm{i}z + 1}{z + \mathrm{i}}. \tag{6.1}$$

The radial coordinate r is directly related to the hyperbolic distance d on the Poincaré half-plane between the arbitrary point (x, y) and the point $\{x = 0, y = 1\}$ which we choose as the initial point of the Brownian motion (3.9):

$$r = \tanh\left(\frac{d}{2}\right). \tag{6.2}$$

The circle at infinity $r = 1$ corresponds to the axis $y = 0$. The unit disk is well suited to study free hyperbolic Brownian motion since it contains explicitly the rotational invariance in the angle θ. In the new coordinates, the metric is

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} = \frac{4}{(1 - r^2)^2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) \tag{6.3}$$

and the Laplace operator is

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{(1 - r^2)^2}{4}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]. \tag{6.4}$$

In the free case, the Fokker–Planck equation for the probability density $Q_t(r, \theta)$ reads

$$\frac{1}{D}\frac{\partial Q}{\partial t} = \frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{(1 - r^2)^2}{4r}Q\right)\right] + \frac{(1 - r^2)^2}{4r^2}\frac{\partial^2}{\partial\theta^2}Q. \tag{6.5}$$

The rules of stochastic calculus [13] give the corresponding stochastic differential equations

$$\text{Itô}\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}t} = D\dfrac{(1 - r^2)^2}{4r} + \sqrt{2D}\dfrac{1 - r^2}{2}\eta_r(t) \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{2D}\dfrac{1 - r^2}{2r}\eta_\theta(t) \end{cases} \tag{6.6}$$

$$\text{Stratonovich}\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}t} = D\dfrac{1 - r^4}{4r} + \sqrt{2D}\dfrac{1 - r^2}{2}\eta_r(t) \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{2D}\dfrac{1 - r^2}{2r}\eta_\theta(t). \end{cases} \tag{6.7}$$

Unlike the system (3.10) for the processes $\{x(t), y(t)\}$, these equations cannot be integrated straightforwardly to give the processes $\{r(t), \theta(t)\}$ as functionals of the white noises $\{\eta_r(t), \eta_\theta(t)\}$. We may, however, use the symmetry of the problem to write the asymptotic probability distribution $Q_\infty(r, \theta)$ in the limit $t \to \infty$ as the uniform measure on the unit circle

$$Q_\infty(r, \theta) = \frac{1}{2\pi}\delta(r - 1). \tag{6.8}$$

For $\mu \neq \frac{1}{2}$, the external constant drift $m = (\mu - \frac{1}{2})$ along the y direction breaks the rotational invariance in the angle θ, and the unit disk is not particularly useful anymore. Nevertheless, one may obtain the asymptotic probability distribution $Q_\infty(r, \theta)$ from the asymptotic law (4.17) for $P_\infty(x, y)$ and from the Jacobian $J\{(r, \theta)/(x, y)\}$,

$$Q_\infty(r, \theta) = \frac{4r}{(r^2 + 1 - 2r\sin\theta)^2}P_\infty\left(\frac{2r\cos\theta}{r^2 + 1 - 2r\sin\theta}, \frac{1 - r^2}{r^2 + 1 - 2r\sin\theta}\right) \tag{6.9}$$

obtaining after some algebra as the generalization of (6.8)

$$Q_\infty(r,\theta) = \delta(r-1)\frac{1}{\sqrt{\pi}}\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu)}\frac{1}{2^{\mu+\frac{1}{2}}}(1-\sin\theta)^{\mu-\frac{1}{2}}. \qquad (6.10)$$

7. Conclusion

We have discussed the exponential functional τ_t which governs transport properties of one-dimensional classical diffusion in a random Brownian potential in terms of hyperbolic Brownian motion. As explained in the introduction, this may be considered as an example of a more general relationship that exists between one-dimensional, or quasi-one-dimensional, disordered systems with a multiplicative stochastic stucture, and diffusion on symmetric spaces.

It is interesting that the time-relaxation spectrum found for the probability distributions $\psi_t(\tau)$ and $X_t(x)$, equations (4.11) and (4.12), also appears in the quantum spectrum of a particle in a constant magnetic field B on the hyperbolic plane [15]. For the hyperbolic geometry, a constant magnetic field is defined as the flux through a covariant surface element $\mathrm{d}S = (1/y^2)\,\mathrm{d}x\,\mathrm{d}y$. The two spectra coincide if we identify the magnetic field $B > 0$ in terms of the drift μ as

$$B = \frac{1+\mu}{2}.$$

In this context the existence of bound states in a strong enough magnetic field $B > \frac{1}{2}$ corresponds to the presence of closed classical orbits [15]. It would be useful to understand this correspondence between such spectra at the level of the stochastic processes themselves.

As a final remark we mention that the Liouville Hamiltonian that we encountered in the path-integral formalism (5.3), and which is closely related to hyperbolic geometry, also appears in the study of some fine properties of one-dimensional quantum localization for the Schrödinger Hamiltonian $H = -\mathrm{d}^2/\mathrm{d}x^2 + V(x)$ where $V(x)$ is a Gaussian white-noise potential [16]. Kolokolov used a path-integral method to compute correlation functions of eigenstates and distribution functions of inverse participation ratio in the high-energy limit. In this formalism the Liouville Hamiltonian shows up in the effective action of the path integral. An expansion of this path integral in a basis of eigenstates then gives expressions very similar to the one we obtained for the probability distribution of the joint law of the processes $\{\tau_t, U(t)\}$.

Acknowledgments

We are very grateful to Marc Yor, whose remark on the connection between exponential functionals of Brownian motion of parameter $\mu = \frac{1}{2}$ and free hyperbolic Brownian motion [11] is at the origin of this work. We also wish to thank him for having kindly given us his papers and other mathematical references related to this subject. We also thank Eugene Bogomolny for interesting discussions.

Appendix. Path-integral method to prove the identity in law (3.20)

In section 3 we derived the identity in law (3.17) mentioned by Yor for the free hyperbolic Brownian motion [11]. As explained before, this identity can easily be generalized to any μ by the same method. Let us now derive it through a path-integral method using an appropriate stochastic reparametrization of time in the path integral. This tool has already proven to be very useful in other contexts [17].

The integration of stochastic differential equations (3.27) gives the process $x(t)$ as the following functional of the two white noises $\{\eta_1, \eta_2\}$:

$$x(t) = \sqrt{2D}\int_0^t \mathrm{d}v\, \eta_1(v) \exp\left[-2D\mu v + \sqrt{2D}\int_0^v \eta_2(s)\mathrm{d}s\right]. \tag{A.1}$$

The marginal law $X_t(x)$ of the process $x(t)$ therefore reads by definition

$$X_t(x) = \int \mathcal{D}\eta_1(u)\mathcal{D}\eta_2(u) \exp\left[-\tfrac{1}{2}\int_0^t \mathrm{d}u\, (\eta_1^2(u) + \eta_2^2(u))\right] \\ \times\delta\left(x - \sqrt{2D}\int_0^t \mathrm{d}v\, \eta_1(v) \exp\left[-2D\mu v + \sqrt{2D}\int_0^v \eta_2(s)\,\mathrm{d}s\right]\right). \tag{A.2}$$

Let us first change from $\eta_2(u)$ to $U(u)$ defined by $\beta U(v)/2 = -2D\mu v + \sqrt{2D}\int_0^v \eta_2(s)\,\mathrm{d}s$:

$$X_t(x) = \int \mathcal{D}\eta_1(u) \exp\left[-\tfrac{1}{2}\int_0^t \mathrm{d}u\, \eta_1^2(u)\right] \\ \times \int_{U(0)=0} \mathcal{D}U(u) \exp\left[-\frac{1}{4D}\int_0^t \mathrm{d}u\, \left(\frac{\beta}{2}\frac{\mathrm{d}U}{\mathrm{d}v} + 2D\mu\right)^2\right] \\ \times\delta\left(x - \sqrt{2D}\int_0^t \mathrm{d}v\, \eta_1(v)\mathrm{e}^{\beta U(v)/2}\right) \tag{A.3}$$

and now from $\eta_1(u)$ to $x(u) \equiv \sqrt{2D}\int_0^u \mathrm{d}v\, \eta_1(v)\mathrm{e}^{\beta U(v)/2}$:

$$X_t(x) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}x(u) \int_{U(0)=0} \mathcal{D}U(u) \exp\left[-\frac{1}{4D}\int_0^t \mathrm{d}u\, \left(\frac{\beta}{2}\frac{\mathrm{d}U}{\mathrm{d}v} + 2D\mu\right)^2\right] \\ \times \exp\left[-\frac{1}{4D}\int_0^t \mathrm{d}u\, \left(\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{e}^{-\beta U(u)/2}\right)^2\right]. \tag{A.4}$$

Let us now perform a time reparametrization of the trajectories $x(u)$ in order to recover the Wiener measure

$$\int_0^t \mathrm{d}u\, \left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 \mathrm{e}^{-\beta U(u)} = \int_0^\tau \mathrm{d}s\, \left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2$$

where

$$\mathrm{d}s = \mathrm{e}^{\beta U(u)}\,\mathrm{d}u \qquad \text{and} \qquad \tau\{U(u)\} = \int_0^t \mathrm{e}^{\beta U(u)}\,\mathrm{d}u. \tag{A.5}$$

The new final time $\tau\{U(u)\}$ is not fixed anymore, but depends on the realization of random potential $U(u)$. To take into account this constraint, we can insert the identity

$$1 = \int_0^\infty \mathrm{d}\tau\, \delta\left(\tau - \int_0^t \mathrm{e}^{\beta U(u)}\,\mathrm{d}u\right) \tag{A.6}$$

to obtain

$$X_t(x) = \int_0^\infty \mathrm{d}\tau \int_{x(0)=0}^{x(t)=x} \mathcal{D}x(u) \int_{U(0)=0} \mathcal{D}U(u) \exp\left[-\frac{1}{4D}\int_0^t \mathrm{d}u\, \left(\frac{\beta}{2}\frac{\mathrm{d}U}{\mathrm{d}v} + 2D\mu\right)^2\right] \\ \times \exp\left[-\frac{1}{4D}\int_0^\tau \mathrm{d}s\, \left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2\right]\delta\left(\tau - \int_0^t \mathrm{e}^{\beta U(u)}\,\mathrm{d}u\right). \tag{A.7}$$

Let us now perform the Gaussian path integral on $x(u)$,

$$\int_{x(0)=0}^{x(t)=x} \mathcal{D}x(u) \exp\left[-\frac{1}{4D}\int_0^\tau \mathrm{d}s \left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2\right] = \frac{1}{\sqrt{4\pi D\tau}} \mathrm{e}^{-x^2/4Dt} \tag{A.8}$$

to get

$$X_t(x) = \int_0^\infty \mathrm{d}\tau \, \frac{1}{\sqrt{4\pi D\tau}} \mathrm{e}^{-x^2/4Dt} \psi_t(\tau) \tag{A.9}$$

where

$$\psi_t(\tau) = \int_{U(0)=0} \mathcal{D}U(u) \exp\left[-\frac{1}{4D}\int_0^t \mathrm{d}u \left(\frac{\beta}{2}\frac{\mathrm{d}U}{\mathrm{d}v} + 2D\mu\right)^2\right] \delta\left(\tau - \int_0^t \mathrm{e}^{\beta U(u)}\,\mathrm{d}u\right) \tag{A.10}$$

is by definition the probability distribution of the functional $\tau_t = \int_0^t \mathrm{e}^{\beta U(u)}\,\mathrm{d}u$. Equation (A.9) is just the translation in terms of probability distributions of the identity in law between the process $x(t)$ and a linear Brownian motion of stochastic time τ_t:

$$x(t) = \sqrt{2D} \int_0^{\tau_t} \mathrm{d}u \, \eta(u).$$

References

[1] Gertsenshtein M E and Vasil'ev V B 1959 *Theor. Prob. Appl.* **4** 391
Karpelevich F I, Tutubalin V N and Shur M G 1959 *Theor. Prob. Appl.* **4** 399
Papanicolaou G C 1971 *SIAM J. Appl. Math.* **21** 13

[2] Dorokhov O N 1982 *JETP Lett.* **36** 318; 1983 *Sov. Phys.–JETP* **58** 606
Mello P A, Pereyra P and Kumar N 1988 *Ann. Phys.* **181** 290

[3] Hüffmann A 1990 *J. Phys. A: Math. Gen.* **23** 5733
Beenaker C W J and Rejaei B 1993 *Phys. Rev. Lett.* **71** 3689
Caselle M 1995 *Phys. Rev. Lett.* **74** 2776, cond-mat 9506024

[4] Balazs N and Voros A 1986 *Phys. Rep.* **143** 109
Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (Berlin: Springer)

[5] Chernov A A 1967 *Biophys.* **12** 336
Solomon F 1975 *Ann. Prob.* **3** 1
Kesten H, Koslov M and Spitzer F 1975 *Compositio Math.* **30** 145
Sinaï Y A G 1982 *Theor. Prob. Appl.* **XXVII** 256
Derrida B and Pomeau Y 1982 *Phys. Rev. Lett.* **48** 627
Derrida B 1983 *J. Stat. Phys.* **31** 433
Kawazu K and Tanaka H 1993 *Sem. Prob.* **XXVII**

[6] Derrida B and Hilhorst H J 1983 *J. Phys. A: Math. Gen.* **16** 2641
de Callan C, Luck J M, Nieuwenhuizen Th and Petritis D 1985 *J. Phys. A: Math. Gen.* **18** 501
Luck J M 1992 *Systèmes désordonnés unidimensionnels* Collection Aléa Saclay

[7] Bouchaud J P and Georges A 1990 *Phys. Rep.* **195** 127
Bouchaud J P, Comtet A, Georges A and Le Doussal P 1990 *Ann. Phys.* **201** 285
Georges A 1988 *Thèse d'état* Université Paris 11

[8] Burlatsky S F, Oshanin G H, Mogutov A V and Moreau M 1992 *Phys. Rev.* A **45** 6955
Oshanin G, Mogutov A and Moreau M 1993 *J. Stat. Phys.* **73**
Oshanin G, Burlatsky S F, Moreau M and Gaveau B 1993 *Chem. Phys.* **177** 803

[9] Monthus C and Comtet A 1994 *J. Physique* I **4** 635

[10] Dufresne D 1990 *Scand. Act. J.* 39
de Schepper A, Goovaerts M and Delbaen F 1992 *Ins. Math. Eco.* **11** 291
Geman H and Yor M 1993 *Math. Fin.* **3** 349

[11] Yor M 1992 *Adv. Appl. Prob.* **24** 509

[12] Yor M 1993 *Ins. Math. Econ.* **13** 23

[13] Gardiner C W 1990 *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* 2nd edn (Berlin: Springer)

[14] Terras A 1985 *Harmonic Analysis on Symmetric Spaces and Applications I* (Berlin: Springer)
[15] Comtet A 1987 *Ann. Phys.* **173** 185
Avron J E and Pnueli A 1992 *Ideas and Methods in Quantum and Statistical Physics* vol 2 (Cambridge: Cambridge University Press)
[16] Kolokolov I 1993 *Sov. Phys.–JETP* **76** 1099; 1994 *Europhys. Lett.* **28** 193
[17] Duru I H and Kleinert H 1979 *Phys. Lett.* **84B** 185
Blanchard P and Sirugue M 1981 *J. Math. Phys.* **22** 1372
Duru I H 1983 *Phys. Rev.* D **28** 2689
Young A and De Witt-Morette C 1986 *Ann. Phys.* **169** 140
Khandekar D C, Lawande S V and Bhagwat K V 1993 *Path Integral Methods and Their Applications* (Singapore: World Scientific)